Algebraic Thinking and Pictorial Growth Patterns

Dorothy, a second grader at an at-risk school where 87 percent of the students receive free or reduced lunches, watched intently as Mrs. T laid out pattern blocks to create a sequence of figures shaped like people (see fig. 1). When Mrs. T asked Dorothy, “What will the next one look like?” Dorothy replied, “Well, if you wanted to go from [the first person] all the way to four … you would put a head, then the body, hands, then the feet. And then you would put a hat, and another hat, and another one, and another one …. I think that’s heavy for a person.” Dorothy thought the hats looked a bit top-heavy; nevertheless, she extended the pictorial growth pattern by correctly building the fourth person (see fig. 1).

Mrs. T then asked Dorothy to build the next person in the sequence. Dorothy did so quickly, using the same strategy she employed to build the fourth person: building the “person” first (which, she said, comprised head, body, hands, and feet) and adding five hats. When Mrs. T asked Dorothy how she knew how to build this fifth person, Dorothy replied, “If that’s four [pointing to the fourth figure] and you wanted to go to five, you would just make the same person and then get one more [hat] from four and put it on five.”

Next, Mrs. T asked Dorothy how many pattern blocks she would need to build the sixth person in the sequence. Without building the person, Dorothy reasoned she would need twelve blocks, six for the person, and one more hat than the fifth figure, which made six hats. Mrs. T asked what the tenth person would look like, and without hesitating, Dorothy responded, “The same person [pointing to the bodies of the people she had already built] except with ten hats on top.” Dorothy had never studied pictorial growth patterns before, but in Mrs. T’s problem-solving interview with her, apparently Dorothy was thinking algebraically.

Algebraic Thinking

Principles and Standards for School Mathematics (NCTM 2000) advocates exploring algebraic ideas beginning in prekindergarten and continuing through the high school mathematics curriculum. Because the word algebra is typically equated with a formalized study of symbol systems (Smith 2003), does this mean we need to teach children in the early elementary grades to interpret and solve equations? Fortunately, the answer to the question is no. Research has provided a broader view of algebra, placing emphasis on algebraic thinking, a concept that focuses on ways of generalizing (Smith 2003). In particular, algebraic thinking emphasizes analyzing change, generalizing relationships among quantities, and representing these mathematical relationships in various ways (NCTM 2000). For children in the early elementary grades, building up algebraic reasoning involves “mov[ing] beyond numerical reasoning to more general reasoning about relationship, quantity, and ways of notating and symbolizing” (Yackel 2002, p. 201).

Instead of defining specific algebra content that should be included in early-grades curricula, Yackel (2002) and others (e.g., NCTM and MSEB 1998; Smith 2003) advocate that educators focus on the activities and questions that encourage children to think in ways leading to a development of algebraic reasoning. One of the many approaches to help children generalize and represent relationships is the study of geometric or pictorial growth patterns (Ferrini-Mundy, Lappan, and Phillips 1997; NCTM 2000; Orton, Orton, and Roper 1999).

In this article, we explore how analyzing and...
extending pictorial growth patterns (like Dorothy’s people pattern found in fig. 1) can promote algebraic thinking in young children, ultimately helping them to extend their numerical reasoning to thinking more generally about relationships.

**What Is a Pictorial Growth Pattern?**

A pictorial growth pattern, sometimes called a geometric pattern (NCTM 2000), is a pattern made from a sequence of figures that change from one term to the next in a predictable way. A pictorial growth pattern involves two variables: some quantifiable aspect of a figure (the dependent variable) is coordinated with an indexing or counting system (the independent variable), identifying the position of the figure in the pattern. Often, a dependent variable is linked to the total number of objects comprising the figure, but other aspects of a pictorial growth pattern could be analyzed. For example, in figure 1, the total number of blocks needed to make a person or the number of hats on a person could both be considered dependent variables.

When students study pictorial growth patterns, often they are encouraged to convert the pictorial growth pattern into a number pattern and analyze the number pattern rather than use the physical construction of the pattern (Orton, Orton, and Roper 1999). However, research shows that the pictorial or physical construction of a pattern can be a powerful tool, enabling students to generalize relationships inherent in the pictorial growth pattern (Orton, Orton, and Roper 1999; Steele 2005; Thornton 2001). Although these valuable pictorial growth patterns are featured in reform-based middle-school mathematics curricula, they are not traditionally explored in early elementary grades; pattern exploration in the lower elementary grades typically emphasizes analyzing and extending repeating patterns of objects and numeric patterns (Liljedahl 2004).

Because pictorial growth patterns can provide a rich context for developing algebraic thinking in older students, we decided to investigate how this context could develop and support children’s algebraic thinking in the early elementary grades. More specifically, we examined the strategies and reasoning processes young children use to analyze, extend, and generalize pictorial growth patterns. We collected data from eight students in grades two and three through two separate one-on-one problem-solving interviews (McCarthy 2005; Slater 2005). During the interviews, we asked the children a series of similar questions for each of five pictorial growth patterns. (See fig. 2 for an example of an interview task.)

**Processes children use to analyze and extend pictorial growth patterns**

As we analyzed the thinking of the children in our study, we found they used similar processes (outlined in fig. 3) to describe, analyze, extend, and
begin to generalize the relationships found in the pictorial growth patterns, even though they had not previously studied these types of patterns (Billings and Tiedt 2006).

**Processes using a covariational analysis of change**

The children first engaged in what Smith (2003) calls a “covariation analysis of the pattern,” which involves analyzing how each variable changes apart from the other. In particular, the children primarily focused their analysis on the dependent variable and analyzed the physical construction of the figures as they changed from one figure to the next. First they analyzed how consecutive figures changed and often quantified this change. (Process 1: Analyze change between consecutive figures.) They used this knowledge to employ process 2 and construct the next figure in the sequence by building onto the previous figure. While constructing or describing subsequent figures, many of the children began to simultaneously use process 3 and identify the aspects of a figure that stayed the same and the aspects that changed from one figure to the next and to draw on this knowledge to build subsequent figures in the pattern.

Thinking back to how Dorothy analyzed and extended the people pattern at the beginning of this article, we can observe how she engaged in these three processes. As Mrs. T laid down the figures in the pattern, Dorothy immediately noticed that the number of hats increased by one: “First there was square, then square, square, then square, square, square” (process 1). However, in observing that subsequent persons in the pattern had one more hat than the previous person (what changes), she also noticed that the body of the person was identical from one person to the next (what stays the same, process 3). To build the fourth person, Dorothy made a body identical to the bodies of the first three persons in the pattern and then added one more hat than the third person had on, for a total of four hats (process 2). To build the fifth person, she repeated this process but added one more hat than the fourth person, for a total of five hats. We consider Dorothy to have successfully described and extended this pattern using a covariational analysis.

When the figures in a pictorial growth pattern did not look like a recognizable object, such as the dot patterns in figure 4, some children had difficulty creating subsequent figures. Although these children correctly determined that the number of dots increased by one (process 1) and added one dot to create each subsequent figure (process 2), they did not identify the aspects of the figure that either stayed the same or changed (process 3) and instead arranged the dots into recognizable objects such as letters L or H (for dot patterns 1 and 2 respectively).

For example, when Adam was initially shown dot pattern 1 and asked what he noticed, he described the first figure as “two dots; it looks like two eyes.” Adam continued to describe the next two figures in this sequence, “This one [points to card 2] looks like an L, this one [points to card 3] looks like an even better L.” When asked if he saw a pattern, Adam replied “Yeah, one, two [points to dots in card 1], one, two, three [point to dots in card 2], one, two, three, four [points to dots in card 3].” He used this knowledge to make the fourth figure in this pattern. Adam stated, “Five, it would be five [dots],” and drew the dots to look like an L (see fig. 5). Similarly, after being shown what the configuration of dots should look like for the fourth and fifth cards in this pattern, and then asked to create the eighth card, Adam stated, “This would have seven [dots] [points to the blank card 6], and this one has eight [points to the blank card 7. Pause.] This one [points to the blank card 8] will have nine.” He then proceeded to draw a dot configuration with nine dots that looked like an L (see fig. 5).

Adam correctly identified how the number of dots changed between consecutive figures (processes 1 and 2) but did not recognize how the placement of the dots on the cards in the pattern stayed the same or changed from one figure to the next. Adam reconfigured the dots so they would look like the letter L and continued to place the dots in different L arrangements as he extended this pattern to figures much further in the sequence.
Processes using correspondence analysis

As the children engaged in the covariational process of analyzing how the physical construction of the figures changed, they began to shift to a “correspondence analysis” (Smith 2003) in which they looked for a relationship between the independent and dependent variables. In particular, they indexed the figure number (independent variable) with an aspect of the figure that was changing (Process 4: Index figure number with changing aspect of dependent variable.) After this correspondence was determined, the children established what the “changing” aspect of the figure would look like based on the figure number. They then replicated the aspect of the figure that “stayed the same” to create a figure located further out in the pattern sequence (Process 5: Extend figure to a large letter n.). This correspondence analysis led the children to predict what subsequent figures would look like and shifted their thinking to a more generalized way of viewing the relationship between the variables. For example, Dorothy began to shift her thinking when asked to describe the total number of blocks needed to build the sixth person. She described this quantity as “six plus five … because if there’s six [hats], and you wanted the total, then you should [have] used the five [pointing to the blocks comprising body of the fifth figure] and add[ed] them together.” Here, she indexed the number of hats according to the number of figures or people and continued to use this indexing to think more generally about the relationship. Dorothy then described how to build the tenth person without recursively building the seventh, eighth, and ninth figures. She stated that “[the tenth person would be] the same person [pointing to the bodies of the people she had already built—see fig. 1] except with ten hats on top.”

Figure 4

Questions for Each Dot Pattern
Show the children the first three dot configurations, where each figure (collection of dots) is on a separate note card.
1. Can you tell me about this one?
2. Do you see a pattern?
3. What would the next one look like? Show me. (Have the student draw the figure on a blank note card. If the student doesn’t make the correct fourth figure, tell the student, “This is the fourth shape that I came up with.” Draw the figure in rows, from top to bottom.)
4. Draw the fifth figure for the student on a separate note card. (Draw as described above.)
5. Lay out two blank note cards for the sixth and seventh figures. Ask the student to draw the eighth figure on a note card. If the student draws or uses the sixth and seventh to figure out the eighth, ask her if there is another way to find the eighth figure without making all the figures.
6. Show the student the eleventh figure. Ask her to show you the tenth figure. Ask her to explain how she knew how to make the tenth figure.
7. Ask the student what the twenty-fifth figure would look like:
   a. How many dots would be in the twenty-fifth figure?
   b. How would you explain to your mom how to make the twenty-fifth figure? (If the student is stuck, ask her, “If you knew how many dots were on the twenty-fourth figure, would you be able to tell me how many dots are on the twenty-fifth one?”)
Typically, we noticed a subtle change in the children’s thinking from a covariational to a correspondence analysis when we asked them to extend the pattern to a figure further out in the sequence. For these extensions, we selected figures that would be tedious to build up recursively, such as the tenth or twenty-fifth terms. All of the children could successfully index situations in which the figure number matched the part of the figure that was changing, such as in figure 1, where the figure number (person number) matches the number of hats. However, some of the children could not consistently index or apply this knowledge when the figures in the pattern did not comprise a recognizable image, such as the dot patterns in figure 4.

Some of the children were able to think more generally about relationships in the more abstract pictorial growth patterns. For example, as Carl, a third grader, analyzed dot pattern 2 (see fig. 4), he first engaged in a covariational analysis of the dots. When initially asked, “Can you tell me about this one?” Carl responded, “This one starts out with four [pointing at card 1]. This one [pointing to card 2] adds one and it makes two dots over here [pointing to right side of card] and three dots over here [pointing to left side of card]. This one [pointing to third card] has one dot up here [pointing to top left dot] and keeps these two [pointing to right side of card].” He noticed the number of dots was growing by one from one figure to the next (process 1) and partitioned the figure into two columns, observing that the right hand column of dots stayed the same while the left hand column grew by one dot with each subsequent figure (process 3).

After extending the pattern recursively for a few terms by adding one dot to the top of the left column to build the next figure in the sequence (process 2), Carl began to incorporate a correspondence analysis into his thinking; he indexed the dot-card number with the number of dots in the left hand column (process 4) and in the process, began thinking more generally about the relationships inherent in the pattern. For instance, when asked what the eighth dot card would look like, Carl drew a card that looked like the computer-generated card in figure 6 and explained, “Cause on the fifth one, there’s six [dots]. There’s six on this side [pointing to left hand column of dots on card 5]. There’s one more than the [card] number and then there’s two on this [right hand column] side on all of them except this one [pointing to the first card], so I just had one more than eight on this side [pointing to left hand column], so it equals nine. Then I put two over here [pointing to right hand column].” Carl similarly described the twenty-fifth card: “There would be twenty-six of these [dots] [pointing to the left hand column of dots on the tenth card], and there would be two over on this side [pointing to the right hand column of dots].” When asked, “How would you explain to your teacher how to make the twenty-fifth one?” Carl replied, “I would tell her, there would be one more [dot] than the [card] number [pointing to the left hand column of dots], then look at the other cards to find out what would be on this [right hand column] side.” From Carl’s description, one can see how he used the physical construction of the dots and transitioned to thinking more generally about the relationship between the number of dots and the figure number.

### Teaching Connections: Use Questions to Promote Algebraic Thinking

As Yackel (2002) notes, “Teacher questions, student solutions, and attempts to follow up on either or both are central to how the instructional activity is realized in action” (p. 201). Consequently, the types of questions we ask children about pictorial growth patterns will influence how they analyze patterns. The questions we designed for our interviews encouraged children to analyze the pictorial or physical representation of the growth pattern and prompted them to think about the relationships inherent in the pattern.

By using the physical representation of the figures in a pattern and engaging in the five processes
we describe in this paper, children successfully extended the pattern to other figures in the sequence and began to think about the relationships in the pattern. However, when children based their analysis on the number of objects in a figure (such as in the dot problems) and did not also utilize the physical construction of the figures, they were unable to consistently and successfully extend the patterns. As teachers, we should challenge children to use the physical construction of the growth pattern as they analyze and extend it.

For example, after we created the first few terms in the pattern, we always began the interview with a “posing question” (Goldin 1998) such as, “What do you notice?” This type of question gives children the freedom to observe whatever they want about a pattern and informs the teacher as to whether or not the children actually recognize a pattern of growth. In addition, the teacher gains valuable insight into the types of relationships that children independently identify before prompting. To encourage children to directly utilize and analyze the physical construction of any pictorial growth pattern, ask them either to build figures or describe their structure, as well as to analyze the next few consecutive figures in the sequence (for example, see questions 3–7 in figure 2 and questions 3–6 in figure 4). These types of questions encourage children to engage in a covariational analysis of the pattern as they notice how consecutive figures in a pattern change and then extend the pattern.

Furthermore, since we want to lay the foundation for children to begin thinking more generally about relationships, it is crucial to ask questions about figures located much further out in the pattern sequence—figures that would be cumbersome to build recursively (for example, question 7 in figure 4).

If children have difficulty creating or describing figures, ask questions that encourage analyzing the pattern from a correspondence point of view to help them begin to think about the general relationships in the pattern. Ask questions such as the following:

- “What is changing in each of these figures?”
- “What is staying the same?”
- “What do you notice about figure number ____ and ____” (whatever they have identified as changing)?
- “How can you use this information to build this figure?”

Encourage children to index the figure number with the changing aspect of the figure, thus helping them understand the general relationship between the variables. Some children may also be ready to describe how to create the figure at any position in the sequence, explicitly transitioning their thinking to the generalized relationships inherent in the pattern.

Use a variety of different types of pictorial growth patterns

We discovered that it was easier for children to analyze and extend pictorial growth patterns constructed to look like recognizable figures—such as the ice cream cone and people patterns—rather than more abstract ones, such as the dot patterns. Consequently, when introducing children to algebraic thinking via pictorial growth patterns, it may be wise to begin with patterns comprised of identifiable figures. We conjecture that children had an easier time analyzing what was the same and what was changing in these patterns because the figures could be chunked into recognizable parts. Thus, after extending the pattern recursively, they could transition to indexing the changing part of the pattern and extend the pattern to a figure much farther out in the sequence. After children are comfortable analyzing pictorial growth patterns comprised of identifiable figures, it is essential that teachers challenge and deepen their thinking by providing pictorial growth patterns comprised of more abstract figures.

Concluding Remarks

The children’s thinking reported in this paper illustrates that analyzing and extending pictorial growth patterns provides a meaningful context for young children to think algebraically as they analyze change and think more generally about the relationships inherent in a pattern. By studying and reflecting about how these children, who had no previous experience with pictorial growth patterns, analyzed and extended pictorial growth patterns, we identified various processes they used to extend these types of patterns. Awareness of these processes can assist teachers to intentionally guide children to analyze change and extend and generalize relationships. By asking children to analyze pictorial growth patterns in the ways described in this article, teachers provide a foundational experience in algebraic thinking that is extendable to the upper elementary and middle school grades, where children use symbols to express their generalizations of mathematical relationships.
References


